4. (10 puntos)

Utilizando el método matricial de valores y vectores propios, determinar la solución general del siguiente sistema de ecuaciones diferenciales:

$$\begin{cases} x'(t) = x(t) - y(t) - z(t) \\ y'(t) = x(t) - y(t) \\ z'(t) = x(t) - z(t) \end{cases}$$

$$\bar{X}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\det(A-\lambda I) = \det \begin{pmatrix} 1-\lambda & -1 & -1 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(1+\lambda)^2\right] - \left[(1+\lambda)\right] + \left[-(1+\lambda)\right] = 0$$

$$(1-\lambda)(1+\lambda)^2 - 2(1+\lambda) = 0$$

$$(1+\lambda)\left[(1-\lambda)(1+\lambda) - 2\right] = 0 \Rightarrow (1+\lambda)\left[1-\lambda^2-2\right] = 0$$

$$\Rightarrow (1+\lambda)(-\lambda^2-1) = 0 \Rightarrow (\lambda+1)(\lambda^2+1) = 0 \quad \begin{cases} \lambda+1 = 0 \\ \lambda^2+1 = 0 \end{cases}$$

$$\Rightarrow \boxed{\lambda_1 = -1}, \quad \lambda_{2,3} = \pm i$$

$$\lambda_1 = -1$$

$$E_{\lambda_1=-1} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| (A-\lambda_1 I)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$(A-\lambda_1 I) = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{ccc|c} a & b & c & \\ \hline \end{array}\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} a = 0 \\ b = -c \end{array}$$

$$E_{\lambda_1=-1} = \left\{ \begin{pmatrix} 0 \\ -c \\ c \end{pmatrix} \right\} \Rightarrow v_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \bar{X}_1 = e^{-t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$