# Introduction

One of the major tasks of scientists is to comprehend the physical environment where we live, which is also the scenario of many fascinating and complex phenomena. As a whole, it is very complicated to understand and explain how the components of our environment interact, therefore one needs to divide it into various (sub) systems. Once we have isolated a specific system, the problem a scientist usually addresses is to uncover the laws governing its evolution. If these laws are deterministic, i.e. the system does not change its state "spontaneously" and if the laws governing the system do not change in time, then the system is referred to as autonomous. On the other hand, (non) autonomous systems can be classified into two groups. If all possible states of the system are characterized by points of a set of finite (resp. infinite) dimension, then the system is said to be finite-dimensional (resp. infinite-dimensional). Another classification can be given in terms of the time with respect to which a system changes its state. If the time takes values along the real line , then the system is referred to as continuous-time system. Contrariwise, if the time takes values only on , then the system is called discrete-time system. In this thesis, we will restrict our attention to finite-dimensional, discrete- and continuous-time dynamical systems.

One of the main approaches for studying a dynamical system is building a mathematical model which represents the laws governing the evolution of the system. This task of course is an art itself; models should be, mathematically speaking, simple enough, in such a way that the model can be handled, but it should not be too simple either, such that the real behavior of the system is not reasonably explained by the model. In the present work, we deal with two different kinds of models, i.e.

$$\dot{x}(t) = F(x(t)), \quad (0.1)$$

$$x \leftrightarrow G(x), \quad (0.2)$$

which represent an N-dimensional, continuous-time and discrete-time dynamical system, respectively, where $F : R^N \to R^N$, $G : R^N \to R^N$ are sufficiently smooth functions. An interesting fact is that most of the systems that appear in applications can be satisfactorily modeled only in terms of nonlinear equations, which have proven to be very difficult to solve. This fact has motivated great efforts of mathematicians to develop fascinating techniques for addressing this problem. One method for dealing e.g. with systems of the type of (0.1) is numerical time-integration, which allows us to explore the dynamics of the system by solving an initial value problem.

For this purpose, we may employ one-step methods, which in their simplest form can be written as

$$x_{n+1} = x_n + hF(x_n), \quad n = 0, 1, \ldots, \quad (0.3)$$

with initial value $x_0 := \xi \in {}^N$ and step-size $h > 0$. This schema is the wellknown Euler's method. By the similarity between systems (0.2) and (0.3), it becomes evident that we can view numerical methods as discrete-time systems. By doing this, we can appeal to the already developed machinery for the study of discrete systems, in such a way that we can achieve a deeper insight into the connection between the dynamics of system (0.1) and its discretization (0.3). This approach gave rise to the born of the so-called Numerical Dynamics. Moreover, if we want to know the effect of external, or control parameters on the dynamics of system (0.1), then additional techniques must be employed. This task is widely addressed by Bifurcation Theory, which, combined with Numerical Dynamics, will be the cornerstone of the present work.

The first part of this thesis addresses the problem of explaining the effect of discretization methods on the local bifurcation diagram of a twoparametric, continuous-time dynamical system with a codimension two singularity (a complete list and the corresponding analysis of the possible codimension two singularities can be found in [44]). Two main questions are outlined and, at some extent, answered, i.e., does a one-step method applied to the continuous-time system reproduce by a “discrete version” the codimension two singularity? If this is so, does the discrete point remain at the same position in both, state space, as well as parameter space? The second major question is a natural consequence of a positive answer to the first one, namely, is the bifurcation picture also reproduced (and maybe shifted) by the discretization method? In the present work, we restrict our attention to two concrete cases: Bogdanov-Takens and fold-Hopf singularities. For the Bogdanov-Takens case, we consider a quite general family of one-step methods: implicit Runge-Kutta methods. On the other hand, for the fold-Hopf case we consider general one-step methods, without assuming any particular structure.

In the second part of the thesis, we concentrate on discrete, global phenomena that occur near a codimension two point. In Chapter 1, we will see that a Bogdanov-Takens bifurcation is turned into a 1 : 1 resonance by Runge-Kutta methods. Thus, a natural question that arises is what can be said about the effect of discretization methods on the emanating homoclinic curve? Answering this question is beyond the scope of this thesis, however, we will provide a numerical approach which allows, at least quantitatively, tackling this issue. That is, we will develop a theory-based numerical method for starting the continuation of tangential homoclinic orbits near a 1 : 1 resonance.

This method will allow us to carry out several interesting experiments (cf. Section 2.3), since homoclinic orbits are one of the most fascinating objects of study in the theory of dynamical systems, because their presence leads to nontrivial dynamics.